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Fixing Einstein's equations

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Einstein's theory of general relativity has not only proven to be physically accurate [1], it also sets a standard for mathematical beauty and elegance—geometrically. When viewed as a dynamical system of equations for evolving initial data, however, these equations have a serious flaw: they cannot be proven to be well-posed (except in special coordinates [2–4]). That is, they do not produce unique solutions that depend smoothly on the initial data. To remedy this failing, there has been widespread interest recently in reformulating Einstein's theory as a hyperbolic system of differential equations [5–20]. The physical and geometrical content of the original theory remain unchanged, but dynamical evolution is made sound. Here we present a new hyperbolic formulation that is strikingly close to the space-plus-time ("3+1") form of Einstein's original equations. Indeed, the familiarity of its constituents make the existence of this formulation all the more unexpected. This is the most economical first-order symmetrizable hyperbolic formulation presently known to us that has only physical characteristic speeds, either zero or the speed of light, for all (non-matter) variables. It also serves as a foundation for unifying previous proposals.

The source of the imperfection in Einstein's theory lies in the fact that it is a constrained theory. Physical initial data cannot be freely specified, yet even infinitesimally perturbed data that violate the physical constraints can lead to results so wildly divergent that they spoil the desired smooth dependence on initial data. This is particularly troublesome in numerical evolution where such violations are unavoidable. The lack of well-posedness is also a serious problem when addressing such a basic question as the global nonlinear stability of flat Minkowski spacetime. In fact, the proof of such stability [21] employs the hyperbolic wave equation of [7] which we discuss below. A well-posed formulation of Einstein's equations would also seem to be an essential starting point for the conventional approach to quantum gravity in which one first quantizes the (unconstrained) classical theory and then imposes the constraints.

The desire to simulate numerically the full nonlinear evolution of Einstein's equations in three dimensions, such as in the collision of two black holes (*cf., e.g.,* [22]), has motivated much of the recent effort on hyperbolic formulations: well-posed underlying equations make stable numerical evolution much more likely than otherwise would be the case, and formulations cast in first-order symmetrizable form are especially suited to numerical implementation. In addition, physical characteristic speeds make it easy to impose good boundary conditions, crucial to a successful numerical scheme. To amplify this point, Einstein's equations contain many unphysical ("gauge") variables among its unknowns, and *a priori* they can travel at any speed. A formulation with only physical characteristic speeds has significant advantages because no explicit separation of physical and unphysical degrees of freedom is required. The physical and unphysical variables propagate at the same speeds and there-

fore satisfy boundary conditions on the same characteristic surfaces. This is particularly important, for example, at the horizon of a black hole, which is a characteristic boundary for physical variables but not for unphysical ones, unless the latter propagate at the speed of light.

The following system of thirty equations will be shown to be symmetrizable hyperbolic

$$0 = \hat{\partial}_0 g_{ij} + 2NK_{ij}, \tag{1}$$

$$R_{ij} = -N^{-1}\hat{\partial}_0 K_{ij} + \bar{R}^{(e)}_{ij} - N^{-1}\bar{\nabla}_i\bar{\nabla}_j N(\alpha, g) + KK_{ij} - 2K_{ik}K^k{}_j, \tag{2}$$

$$2g_{ij}R_{k0} = \hat{\partial}_0\bar{\Gamma}_{kij} + \partial_j(NK_{ki}) + \partial_i(NK_{kj}) - \partial_k(NK_{ij}) - 2g_{ij}N\bar{\nabla}^m(K_{km} - g_{km}K) \tag{3}$$

(notation elaborated below). This form suggests the name "Einstein-Christoffel system." It is convenient for pedagogical reasons to replace the third equation by the equivalent equation

$$4g_{k(i}R_{j)0} = \hat{\partial}_0\mathcal{G}_{kij} + \partial_k(2NK_{ij}) - 4g_{k(i}N\bar{\nabla}^m(K_{j)m} - g_{j)m}K) \tag{4}$$

The indices i, j, k, m run over the spatial indices 1, 2, 3, and the Einstein convention of summing over repeated indices is used. A short-hand notation to indicate symmetrization, $A_{(i}B_{j)} = (1/2)(A_iB_j + A_jB_i)$, is used in the previous equation.

To establish notation, assume that spacetime has topology $\Sigma \times R$ with metric given in the foliation-adapted basis,

$$ds^2 = -N^2(dt)^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{5}$$

Here, $N(\alpha, g)$ is the lapse scalar, and $\beta^i(x,t)$ is the spatial shift vector, freely specifiable on the spacelike slices $t =$ constant. The lapse N is determined through $N = \alpha g^{1/2}$, where $\alpha(x,t)$ is a freely specified "slicing" density (of weight -1) and $g = \det g_{ij}$ is the determinant of the spatial metric g_{ij}. The spatial derivatives of the metric are denoted by

$$\mathcal{G}_{kij} = \partial_k g_{ij}. \tag{6}$$

This is a subtle element, as it will transpire that while this relation is imposed initially, it may not hold for the evolved quantities, as discussed below. The spatial Christoffel symbols in this metric are, with $\bar{\Gamma}^m{}_{ij} = g^{mk}\bar{\Gamma}_{kij}$,

$$\bar{\Gamma}_{kij}(\mathcal{G}) \equiv (1/2)(\mathcal{G}_{jki} + \mathcal{G}_{ikj} - \mathcal{G}_{kij}). \tag{7}$$

To focus attention on $\mathcal{G}_{kij}$, the Christoffel symbols will not be used here as independent variables, though they could be, but only as a compact notation for this expression in terms of $\mathcal{G}_{kij}$; the explicit functional dependence "$(\mathcal{G})$" is intended to reflect this choice. Finally, K_{ij} denotes the extrinsic curvature of the slice Σ, and $K = K^k{}_k$ is its trace.

The derivative $\bar{\nabla}_k$ is the spatial covariant derivative operator in Σ. The derivative $\hat{\partial}_0 = \partial_t - \pounds_{\boldsymbol{\beta}}$, where $\partial_t = \partial/\partial t$ and $\pounds_{\boldsymbol{\beta}}$ is the Lie derivative along the shift vector $\boldsymbol{\beta}$ in a $t =$ constant slice, is the natural time derivative for evolving time-dependent spatial tensors. It is the extension to tensors of the (non-coordinate) basis vector $\partial_0 = \partial_t - \beta^k\partial_k$ ($\partial_k = \partial/\partial x^k$) that is normal to the slice Σ. Note that while $[\partial_0, \partial_j] = \partial_0\partial_j - \partial_j\partial_0 = (\partial_j\beta^k)\partial_k \neq 0$,

$$[\hat{\partial}_0, \partial_k] = 0. \tag{8}$$

On the left-hand side of (2) and (3) or (4), R_{ij} and R_{j0} are spacetime Ricci curvature tensors and are to be replaced by their appropriate expressions in terms of matter stress tensors from Einstein's equations, $G_{\mu\nu} \equiv R_{\mu\nu} - (1/2)g_{\mu\nu}R = 8\pi T_{\mu\nu}$. Here, μ, ν run from 0 to 3, $R = R^{\mu}{}_{\mu}$, and $T_{\mu\nu}$ is the matter stress tensor. $\bar{R}_{ij}^{(e)}$ is the spatial Ricci curvature tensor of the spacelike slice Σ. It is essential to manipulate the standard form of $\bar{R}_{ij} = \partial_k \bar{\Gamma}^k{}_{ij} - \partial_j \bar{\Gamma}^k{}_{ik} + \bar{\Gamma}^k{}_{mk} \bar{\Gamma}^m{}_{ij} - \bar{\Gamma}^k{}_{mj} \bar{\Gamma}^m{}_{ik}$ into a distinct but (initially) equivalent form, indicated below, and the superscript "(e)" reflects this change.

Einsteinian initial data for the system (1), (2), (4) are g_{ij}, K_{ij}, $\mathcal{G}_{kij}$, specified on an initial slice Σ_0, and presumed to satisfy the Einstein constraints, $G^0{}_0 = 8\pi T^0{}_0$ and $R^0{}_k = 8\pi T^0{}_k$. This system of initial constraints is well understood as a semilinear elliptic system [24,25]. A mathematically well-posed form of the twice-contracted Bianchi identities [26,27] shows that these initial-value constraints remain satisfied if the equations of motion are equivalent to $R_{ij} = 8\pi(T_{ij} - (1/2)g_{ij}T^{\mu}{}_{\mu})$.

That the system (1), (2), (4) is hyperbolic is not obvious, but its content is easy to grasp. The first equation (1) is simply a definition of the extrinsic curvature K_{ij}. The second equation (2) is the 3+1 decomposition of the space-space components of the spacetime Ricci tensor. As such, these are the basic geometric ingredients of Einstein's original equations and of all 3+1 formulations of general relativity [23–25]. The remarkable fact is that the equation (4) completes the first two into a symmetrizable hyperbolic system. The content of equation (4) is also readily understood.

If one applies $\hat{\partial}_0$ to (6) and uses (8) and (1), one obtains the identity

$$\hat{\partial}_0 \mathcal{G}_{kij} = -\partial_k(2NK_{ij}). \tag{9}$$

This is the right hand side of (4) aside from the 3+1 decomposition of $4g_{k(i}R_{j)0}$. Ordinarily in Einstein's theory,

$$R_{0j} = -N\bar{\nabla}^m(K_{jm} - g_{jm}K) \tag{10}$$

is a constraint—the "momentum" constraint—because it involves no time derivatives. What is special about (4) is that it makes the momentum constraint dynamical by combining it with the identity (9) involving a time-derivative. This defines a modified evolution of $\mathcal{G}_{kij}$ when the constraint is not satisfied, that is, when (10) does not hold after R_{0j} is replaced by its matter expression.

The identity (9) is closely related to metric compatibility of the connection. In a general spatial frame, a connection is metric compatible if and only if $\bar{\Gamma}_{ijk} + \bar{\Gamma}_{jik} = \partial_k g_{ij}$. Taking the time derivative of this condition and applying (4) gives

$$2\hat{\partial}_0 \bar{\Gamma}_{(ij)k} = -\partial_k(2NK_{ij}) + 4g_{k(i}N\bar{\nabla}^m(K_{j)m} - g_{j)m}K) + 4g_{k(i}R_{j)0}. \tag{11}$$

Thus, if the momentum constraint were violated, metric compatibility of $\bar{\Gamma}$ would be lost in the evolution of (11). While $2\bar{\Gamma}_{(ij)k} = \mathcal{G}_{kij}$ always holds, the evolved $\mathcal{G}_{kij}$ would no longer be the spatial derivative ∂_k of the evolved g_{ij}.

To motivate the system (1), (2), (4) further, consider two of its predecessors, the Einstein-Ricci formulation [7,12–14,18] and the Frittelli-Reula formulation [10,16]. The third-order

Einstein-Ricci system consists of (1) and a wave equation built from (2) and (10) through the combination

$$\hat{\partial}_0 R_{ij} - \bar{\nabla}_i R_{j0} - \bar{\nabla}_j R_{i0} = N\hat{\Box} K_{ij} + J_{ij} + S_{ij}. \tag{12}$$

It is called third-order because of the effective number of derivatives of g_{ij} in (12). Here, $\hat{\Box} = -N^{-1}\hat{\partial}_0 N^{-1}\hat{\partial}_0 + \bar{\nabla}^k\bar{\nabla}_k$. J_{ij} is a nonlinear function of K_{ij}, N, their first derivatives, and the second derivatives of N. S_{ij} is a potentially troublesome term involving a second spatial derivative of K and a third derivative of N. The behavior of S_{ij} is tamed by using $N = \alpha(x,t)g^{1/2}$ [7,12] (or by imposing generalized harmonic slicing [18] with a gauge source [4]). Note that the use of α permits any time-slicing to be employed.

This system can be put in first-order form by introducing new variables to represent the temporal and spatial derivatives of K_{ij} and of N. Together with (1) and the equations obtained by applying $\hat{\partial}_0$ to $\bar{\Gamma}^k{}_{ij}$, and using α to eliminate N, a system of 66 equations is found. This system is manifestly spatially covariant, is expressed in 3+1 geometric variables, and has only physical characteristic speeds, either zero or the speed of light.

One may wonder about the large number of equations and about a deeper meaning behind the combination in (12). Regarding the number of equations, the Einstein-Ricci system is equivalent (for Einsteinian initial data) to the Einstein-Bianchi system [19,20] which also has 66 equations. There, it is evident that this number of equations is precisely that needed to incorporate the full Bianchi identities and to propagate the Riemann curvature tensor explicitly in a system having only physical characteristics (otherwise, *cf.* [15]).

The Frittelli-Reula system [10,16], in contrast, has 30 equations, is expressed in non-covariant variables, and admits superluminal characteristic speeds for some (unphysical) degrees of freedom. It was constructed by proposing an energy norm built from the extrinsic curvature and spatial derivatives of the metric. Frittelli and Reula parametrize their construction and find a large family of hyperbolic systems with different characteristics, none wholly physical. Friedrich has observed that an equation for the metric constructed from (1) and (2), while not of known hyperbolic type, has only physical characteristics [15]. A natural question is whether there are further thirty-variable hyperbolic systems and whether any has only physical characteristics. The Einstein-Christoffel system presented here is such a system, and it is easy from it to see how to extend the Frittelli-Reula construction.

The Einstein-Ricci system has only physical characteristics, but its equations number more than twice those of the Frittelli-Reula system. A natural question is whether the third-order form can be put in first-order form to achieve a thirty variable system. The answer is yes. To see why this might be possible, consider the simple wave equation

$$\partial_t^2 u - \partial_x^2 u = 0, \tag{13}$$

This can be put in first order form in two ways. The easiest is to introduce the derivatives of the dependent variable as new variables. Introduce $U = \partial_t u$ and $V = \partial_x u$ to reach the system

$$\begin{aligned} \partial_t u &= U, \\ \partial_t U - \partial_x V &= 0, \\ \partial_t V - \partial_x U &= 0. \end{aligned} \tag{14}$$

The last equation is an integrability condition reflecting the commutativity of the partial derivatives. This parallels the way that the first-order form of the Einstein-Ricci system was obtained from (12).

The second way to get to first order form is to pull apart the wave equation to obtain first order pieces

$$\begin{aligned}\partial_t u - \partial_x v &= 0,\\ \partial_t v - \partial_x u &= 0.\end{aligned} \tag{15}$$

The first method is essentially one derivative higher. It necessarily involves more variables. Note that the wave equation (13) is reconstructed from this system (15) by taking a time derivative of the first equation and adding a spatial derivative of the second. This parallels the structure in (12) that leads to the third-order Einstein-Ricci system. This encourages the speculation that a "pulled-apart" system analogous to (15) is possible. The obstacle however is that the momentum constraint as usually construed is not a dynamical equation, so the obvious "pulled-apart" system is not a hyperbolic system. The key idea is that adding a suitably chosen dynamical identity to the momentum constraint overcomes this obstacle and leads to a symmetrizable hyperbolic system.

To begin, we work with $\mathcal{G}_{kij}$ rather than $\bar{\Gamma}_{kij}$. Focus on the derivatives of the Christoffel symbols contained in $\bar{R}_{ij} - N^{-1}\bar{\nabla}_i\bar{\nabla}_j N$. These are the essential terms from the standpoint of hyperbolicity. [Recall that $N = \alpha g^{1/2}$, so $\bar{\nabla}_j N = g^{1/2}\partial_j\alpha + \bar{\Gamma}^k{}_{jk}(\mathcal{G})g^{1/2}\alpha$.] These terms are reorganized as follows:

$$\begin{aligned}\partial_k\bar{\Gamma}^k{}_{ij}(\mathcal{G}) - \partial_j\bar{\Gamma}^k{}_{ik}(\mathcal{G}) - \partial_i\bar{\Gamma}^k{}_{jk}(\mathcal{G}) = &\\ = \frac{1}{2}\Big(\partial_k[g^{km}(\mathcal{G}_{jmi} + \mathcal{G}_{imj} - \mathcal{G}_{mij})] - \partial_j(g^{rs}\mathcal{G}_{irs}) - \partial_i(g^{rs}\mathcal{G}_{jrs})\Big)&\\ = \frac{1}{2}\Big(-\partial_k(g^{km}\mathcal{G}_{mij}) + \partial_i[g^{rs}(\mathcal{G}_{rsj} - \mathcal{G}_{jrs})] + \partial_j[g^{rs}(\mathcal{G}_{rsi} - \mathcal{G}_{irs})]&\\ +g^{kr}g^{sm}(\mathcal{G}_{irs}\mathcal{G}_{kmj} + \mathcal{G}_{jrs}\mathcal{G}_{kmi}) - g^{kr}g^{sm}\mathcal{G}_{krs}(\mathcal{G}_{jmi} + \mathcal{G}_{imj})\Big),&\end{aligned} \tag{16}$$

where $\partial_k(g^{km}\mathcal{G}_{jmi}) = \partial_j(g^{km}\mathcal{G}_{kmi}) + g^{kr}g^{sm}(\mathcal{G}_{jrs}\mathcal{G}_{kmi} - \mathcal{G}_{krs}\mathcal{G}_{jmi})$ has been used. Introducing

$$\begin{aligned}f_{kij} &\equiv \frac{1}{2}[\mathcal{G}_{kij} - g_{ki}g^{rs}(\mathcal{G}_{rsj} - \mathcal{G}_{jrs}) - g_{kj}g^{rs}(\mathcal{G}_{rsi} - \mathcal{G}_{irs})]\\ &= \bar{\Gamma}_{(ij)k} + \frac{1}{2}g_{ki}g^{rs}(\bar{\Gamma}_{rsj} - \bar{\Gamma}_{jrs}) + \frac{1}{2}g_{kj}g^{rs}(\bar{\Gamma}_{rsi} - \bar{\Gamma}_{irs})\end{aligned} \tag{17}$$

puts the leading derivatives of (2) in the familiar form

$$R_{ij} = -N^{-1}\hat{\partial}_0 K_{ij} - \partial^k f_{kij} + l.o._{ij}, \tag{18}$$

where $l.o._{ij}$ stands for lower order terms containing no derivatives of unknowns. They are

$$\begin{aligned}l.o._{ij} = HK_{ij} - 2K_{ik}K^k{}_j - \alpha^{-1}\bar{\nabla}_i\partial_j\alpha - \bar{\Gamma}^k{}_{jk}(\mathcal{G})\alpha^{-1}\bar{\nabla}_i\alpha&\\ +2\bar{\Gamma}^k{}_{mk}(\mathcal{G})\bar{\Gamma}^m{}_{ij}(\mathcal{G}) - \bar{\Gamma}^k{}_{mj}(\mathcal{G})\bar{\Gamma}^m{}_{ik}(\mathcal{G}) - g^{kr}g^{sm}\mathcal{G}_{krs}f_{mij}&\\ +\frac{1}{2}[g^{kr}g^{sm}(\mathcal{G}_{jrs}\mathcal{G}_{kmi} + \mathcal{G}_{irs}\mathcal{G}_{kmj}) - g^{kr}g^{sm}\mathcal{G}_{krs}(\mathcal{G}_{jmi} + \mathcal{G}_{imj})].&\end{aligned} \tag{19}$$

Turn to consider (4). From (17), one computes

$$g_{ki}R_{j0} + g_{kj}R_{i0} = -\hat{\partial}_0 f_{kij} - \partial_k(NK_{ij}) + l.o._{kij}. \tag{20}$$

The lower order terms are

$$\begin{aligned} l.o._{kij} = {} & NK_{ki}g^{rs}(\mathcal{G}_{rsj} - \mathcal{G}_{jrs}) + NK_{kj}g^{rs}(\mathcal{G}_{rsi} - \mathcal{G}_{irs}) \\ & + g_{ki}[K_{mj}\partial^m N - H\partial_j N + Ng^{rs}\bar{\Gamma}^k{}_{jr}(\mathcal{G})K_{sk} - N(\mathcal{G}_{rsj} - 2\mathcal{G}_{jrs})K^{rs}] \\ & + g_{kj}[K_{mi}\partial^m N - H\partial_i N + Ng^{rs}\bar{\Gamma}^k{}_{ir}(\mathcal{G})K_{sk} - N(\mathcal{G}_{rsi} - 2\mathcal{G}_{irs})K^{rs}]. \end{aligned} \tag{21}$$

The subsystem (18) and (20) [completed by (1)] is obviously symmetrizable hyperbolic because it has the familiar structure of a wave equation in first order form. [It is not rigorously symmetric hyperbolic because a metric is present to raise the spatial derivative index in (18).] It is also clear that to build a wave equation in K_{ij} from (18) and (20), one forms the combination in (12). This reveals the meaning behind this combination. The characteristic speed in the subsystem (18), (20) is the speed of light, so the extrinsic curvature and the connection propagate at the speed of light. From (1), the metric propagates at speed zero.

It should be emphasized that the wave equation obtained from (18), (20) is not exactly the same as the third-order Einstein-Ricci system. They differ by lower order terms proportional to constraints. This means that they will agree for Einsteinian initial data, but may disagree when the constraints are violated.

The energy norm for the system (1), (18), (20) is the integral over Σ of $K^{ij}K_{ij} + f^{kij}f_{kij}$, where $f^{kij} = g^{km}g^{ir}g^{js}f_{mrs}$. When the energy norm is expressed in terms of $P_{ij} = K_{ij} - g_{ij}K$, $P = P^k{}_k$, and $M_{kij} = -(1/2)(\mathcal{G}_{kij} - g_{ij}g^{rs}\mathcal{G}_{krs})$, the result can be compared to the *ansatz* of Frittelli-Reula [16]. Additional terms are present beyond those that they considered. Their energy norm *ansatz* can be generalized by adding arbitrary positive multiples of quadratic terms formed from traces of P_{ij} and of M_{ijk}. Similarly, their parametrization of M_{ijk} can be extended using combinations of traces of $\mathcal{G}_{ijk}$, such as occur in f_{kij}. This produces a larger many-parameter family of symmetrizable hyperbolic systems equivalent to Einstein's equations. Some of these other systems also have only physical characteristics, for example, a multiple of the Hamiltonian constraint $G^0{}_0$ can be added to (2) if a compensating multiple of the momentum constraint R_{0k} is added to (3) and (4).

The Einstein-Christoffel system discussed here is a well-posed system of 30 equations that has only physical characteristic speeds and can be expressed in 3+1 geometric variables. Spatial covariance of the system is not explicit, but present nonetheless. This formulation clarifies the relationships among Einstein's original equations and the Einstein-Ricci and Frittelli-Reula hyperbolic formulations. One can see further links to the Einstein-Bianchi, Friedrich [5,6,8,15] and Bona-Massó [9,11] formulations. It will be very interesting to implement this system numerically to see how it behaves. Having only physical characteristics should prove very useful when imposing boundary conditions on the horizon of a black hole since no information, physical or otherwise, can leave the black hole, and all information will enter the black hole in a physical way.

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